Chapter 9
Wave Interference and Modes in Random Media

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9.1 Introduction

Mass and energy are transported via waves. These waves are quantum mechanical for classical particles, such as electrons, and classical for quantum mechanical particles, such as photons. For particles that do not mutually interact, transport through disordered media reduces to the study of wave scattering by inhomogeneities in the phase velocity of a medium. Examples of disturbances within uniform or periodic media are atomic dislocations in resistors, molecules in the atmosphere, or dielectric fluctuations in composite media. When the wave is multiply scattered, but scattering is sufficiently weak that the wave returns only rarely to a coherence volume within the sample through which it has passed, average transport may be described by particle diffusion. However, the wave nature is still strongly exhibited in fluctuations on a wavelength scale and in the statistics of transport. Though details of the scattering process depend on the type of waves and the specific environment, many essential characteristics of wave transport on length scales greater than both the wavelength $\lambda$ and the transport
mean free path $\ell$ are strikingly similar. Thus, for example, in the limit of particle diffusion, electronic conductance follows Ohm’s law, and optical transmission through clouds and white paint correspondingly falls inversely with thickness. In this review, we consider features of multiply scattered waves in samples in which the wave is temporally coherent throughout the sample. This is readily achieved for classical wave scattering from static dielectric structures, but is only obtained for electrons interacting with restless atoms at ultralow temperatures in mesoscopic samples intermediate in size between the microscopic atomic scale and the macroscopic scale.

The superposition of randomly scattered waves in static disordered systems produces a random spatial pattern of field or intensity referred to as the speckle pattern because of its grainy appearance, as shown schematically in Fig. 9.1. The speckle pattern at different frequencies provides a complex fingerprint of the interaction of a wave with the sample. However, in general, it is not possible to infer the internal structure of a body, even from a complete set of such patterns for all incident wave vectors. Indeed, even the forward problem of calculating the speckle pattern from a given structure for three-dimensional systems cannot be solved at present, except for samples with dimensions considerably larger than the wavelength scale. Nonetheless, essential elements of a description of wave transport can be inferred from the statistics of the speckle pattern of radiation scattered from ensembles of random samples. Here, we examine the statistics of random transmission variables within a random ensemble of sample configurations, as well as the statistics of the evolution of the speckle pattern as a whole.

Figure 9.1 Schematic diagram of speckle pattern produced by multiple scattering in a random medium.
The statistics of wave transport depend on the spatial distribution of excitation within the sample. Waves may be either exponentially localized within the sample or may extend throughout the sample. The localization transition is a consequence of the change in the nature of modes of classical and quantum mechanical waves in disordered systems and engenders a dramatic change in transport. Wave propagation and localization may be characterized by a host of interrelated parameters expressing different aspects of propagation. Propagation parameters include (1) the degree of spectral overlap of modes of the random medium, which is the ratio of the averages of the spectral width and spacing of the modes \( \delta = \delta / \Delta \nu \), (2) the dimensionless conductance \( g \), which is the sum of transmission coefficients over all incident and outgoing transverse propagation channels, \( a \) and \( b \), respectively, (3) the variance of total transmission for a single incident channel normalized by the average value of the transmission \( \text{var}(s_a) = T_a / \langle T_a \rangle \), (4) the degree of intensity correlation between points for which the field correlation function vanishes \( \kappa \), and (5) the probability of return of a partial wave to a coherence volume within the sample \( P_{\text{return}} \). These parameters are related as follows:

\[
\delta = g = 2/3 \text{var}(s_a) = 2/3 \kappa = 1/P_{\text{return}}.
\]  

The nature of propagation within the sample is seen both in fluctuations of random variables of the scattered wave and in the evolution of the scattered wave with shift in incident frequency. The probability distributions of total transmission and of various measures of transformation of the speckle pattern with frequency shift are described by a single functional form that depends only on the variance of the corresponding distribution. Characteristics of wave propagation in random media are discussed from two perspectives: one emphasizing the interference of partial waves within the medium, the other focusing on the underlying modes excited within the medium.

In this review, we consider this divide from two perspectives with some illustrative examples. We first consider the complex interference within the medium of multiply scattered partial waves. The interference of these waves leads to weak localization, which suppresses transport. We then treat transport by considering the different spatial, spectral, and temporal characters of the electromagnetic modes of the medium. These approaches are closely related, as it is the interference of waves that produces the field distribution in a given mode.

### 9.2 Wave Interference

#### 9.2.1 Weak localization

Average transport from any region within a random sample is suppressed by constructive interference of waves following paths that return to a coherence volume through which the waves passed and which differ only in the sense in which they are traversed. The coherent sum of the complex amplitudes for return for two time-reversed paths may be written as \( A = A_0 + A_\varphi \). Because the
amplitudes and phase associated with these partial waves are identical, the probability of return is proportional to

$$|A|^2 = |A_\varphi + A_\psi|^2 = |2A_\varphi|^2 = 4|A_\varphi|^2,$$  \hspace{1cm} (9.2)

which corresponds to twice the return probability obtained for the incoherent sum $|A_\varphi|^2 + |A_\psi|^2 = 2|A_\varphi|^2$. In samples in which the probability of return is small, average transport is not appreciably affected by wave interference. The average over an ensemble of random configurations of the intensity at a point is then proportional to the weighted average of the amplitude squared for each of the sinuating partial waves that passes through the point. The probability of return to a point for an ensemble of random configurations is then well approximated by the diffusion equation for intensity or photon density. However, when the probability of return approaches unity, wave transport is strongly suppressed by wave interference and the wave becomes exponentially localized within the medium. The limits in which the particle diffusion model holds can therefore be established by considering the probability of return of a random walker to a coherence volume in the medium. Since the actual probability of return to a coherence volume $P_{\text{return}}$ is twice the probability for the incoherent addition of the associated paths

$$P_{\text{return}} = 2P_{\text{inc}}^{\text{return}},$$  \hspace{1cm} (9.3)

the enhanced return due to wave interference, known as weak localization or coherent backscattering, is equal to $P_{\text{inc}}^{\text{return}}$. This corresponds to a reduction in transport and to a suppression of the bare or Boltzmann diffusion coefficient $D_B$ to a renormalized effective diffusion coefficient $D$. In the limit $P_{\text{return}} \ll 1$, the fractional reduction in the diffusion coefficient is given by

$$\frac{D_B - D}{D_B} = P_{\text{inc}}^{\text{return}}.$$  \hspace{1cm} (9.4)

Here, $D_B = v\ell / d$ is the photon diffusion coefficient in the Boltzmann approximation in which interference of waves returning to a point is neglected and $d$ is the dimensionality of the sample. We take the coherence volume to be $V_c \sim (\lambda/2)^d$, where $\lambda/2$ is the field correlation length that corresponds to the first zero of the field correlation function within the medium.

In an unbounded medium, $P_{\text{inc}}^{\text{return}}$ may be calculated in the diffusion model, and is given by

$$P_{\text{inc}}^{\text{return}} = \frac{V}{\tau_c} \int_0^{\infty} P(0,t)dt.$$  \hspace{1cm} (9.5)
Here,

\[ P(r, t) = \frac{1}{(4\pi D_g t)^{d/2}} \exp\left(\frac{-r^2}{4D_g t}\right) \]  

(9.6)

is the Green’s function of the diffusion equation, and \( \tau_c \) is the time needed to travel a coherence length \( \tau_c = \lambda/2v \), where \( v \) is the transport velocity. The lower limit of the integral is the earliest time of return, which is twice the mean free time between scattering events \( \tau = \ell / v \). This corresponds to the time to return for a single scattering event at a distance of a single mean free path from the point \( r = 0 \), giving

\[ P_{\text{return}}^{\text{inc}} = \frac{(\lambda/2)^d}{\tau_c(4\pi D_g)^{d/2}} \int_0^\infty \frac{dt}{t^{1/2}}. \]  

(9.7)

which is finite only for \( d > 2 \).\(^{34,60}\) \( P_{\text{return}}^{\text{inc}} \) is unbounded in one- or two-dimensional media. This suggests that escape from a point is substantially suppressed due to coherent backscattering and that the intensity at the origin may not vanish with increasing delay. Thus, the wave is localized independently of the strength of scattering for \( d \leq 2 \). Only above the marginal dimension for localization \( d = 2 \), may \( P_{\text{return}} \) be less than unity. For \( d > 2 \),

\[ P_{\text{return}} = \frac{(\lambda/2)^d}{\tau_c(4\pi D_g)^{d/2}} \frac{1}{d/2 - 1} (2\tau)^{1-d/2}. \]  

(9.8)

The interference of waves returning to a point then results in a reduction of \( D \) from its Boltzmann value \( D_B \). Expressing \( D_B \) and \( \tau \) in terms of the mean free path for \( d = 3 \) gives

\[ P_{\text{return}} = \frac{D_B - D}{D_B} = \frac{3\sqrt{6\pi}}{8} \frac{1}{(k\ell)^2} = \frac{1.63}{(k\ell)^2}. \]  

(9.9)

Thus, \( P_{\text{return}} \approx 1 \) when \( k\ell \approx 1 \). The influence of interference is then so strong that \( D \) tends to zero and waves are localized. Thus, \( k\ell \approx 1 \) is the threshold for electron localization transition in 3D, which is the Ioffe-Regel criterion.\(^{26}\) Once \( \ell < \lambda/2\pi \), the phase of the wave does not change substantially between scattering events, and it is not possible to define a trajectory with a specific direction. The particle diffusion model can therefore no longer describe transport. Thus, limitations on transport are not exclusively the province of quantum
mechanics, but arise from the wave nature of the electron. This stands in contrast to the original quantum mechanical perspective of the Anderson model. In that model, an electron interacts with an atomic lattice with disorder in either the diagonal or off-diagonal elements of the Hamiltonian, which represent the site energy and the coupling coefficients, respectively. Anderson showed that a transition in the nature of propagation may occur for \( d = 3 \) as a function of electron energy or scattering strength. At the mobility edge separating diffusive and localized waves, the correlation length diverges with a critical exponent as the transition is approached from either the diffusive or localized side of the mobility edge. \(^{28,29,32,64–66}\)

### 9.2.2 Coherent Backscattering

The impact of coherent backscattering can be directly observed as an enhancement of retroreflected light from a random medium.\(^{52–55,60,61}\) The trajectories of two partial waves that follow time-reversed paths within the sample are then scattered at an angle \( \theta \) to the normal, as shown in Fig. 9.2. These waves interfere in the far field. When scattering is time-reversal invariant, the accumulated phase shifts for waves traversing the same path within the medium but in opposite senses are the same. However, a phase difference arises, due to the different lengths of the trajectories outside the sample

\[
\Delta \phi = \frac{2\pi}{\lambda} \Delta r = k \cdot \rho, \quad (9.10)
\]

where \( \Delta r \) is the additional pathlength outside the sample and \( \rho \) is the transverse excursion of the wave along the sample surface. The contribution to the field at a point \( r \) in the far-field associated with the partial waves that enters the sample normally at \( r_i \) and emerges from the sample at \( r_i + \rho \), after following a trajectory \( \alpha \) within the sample between these points, and which is then scattered with wave vector \( k \), is the real part of the complex field

\[
p(r_i, \alpha, \rho, k) e^{i\phi} e^{ik[r-(\tau+\rho)]}. \quad (9.11)
\]

If we add the fields for the time-reversed pair of partial waves shown in Fig. 9.2, we obtain

\[
p(r_i, \alpha, \rho, k) e^{i\phi} e^{ik[r-(\tau+\rho)]} (1 + e^{i\Delta \phi}). \quad (9.12)
\]

Summing all pairs gives the intensity in the direction of \( k \) at an angle \( \theta \) to the normal

\[
I(\theta) = \left| \sum_\alpha \sum_{\tau,\rho} p(r_i, \alpha, \rho, k) e^{i\phi} e^{ik[r-(\tau+\rho)]} (1 + e^{i\Delta \phi}) \right|^2, \quad (9.13)
\]
Figure 9.2 Schematic of coherent backscattering for a pair of partial waves traversing along path $\alpha$ in opposite directions.

where the prime indicates the sum is taken over pairs with reversed points of entry and exit. The average over an ensemble of random configurations, denoted by $\langle \ldots \rangle$, can be obtained by averaging scattered light over time in a colloidal sample or by spinning or translating a static sample. Cross terms in the square of the sum of the partial waves for different paths do not contribute to the ensemble average, since the phase accumulated within the sample for different configurations is not correlated. This gives

$$\langle I(\theta) \rangle = \langle \sum_\alpha \sum_{\rho \rho'} P(r, \alpha, \rho, k)(2 + 2\cos(k \cdot \rho)) \rangle = I_{\text{inc}} + I_c, \quad (9.14)$$

where

$$P(r, \alpha, \rho, k) = p^2(r, \alpha, \rho, k), \quad (9.15)$$

and $I_{\text{inc}}$ and $I_c$ are the intensity values associated, respectively, with the incoherent and coherent sums over paths. The factor of 2 appears because the sum is taken over pairs of time-reversed paths. In the exact backscattered direction, $k \cdot \rho = 0$, and the intensity is double that of the incoherent background. This enhancement is seen in the measurement of coherent backscattering shown in Fig. 9.3. The enhanced reflection corresponds to a reduction in transmission associated with a renormalization of $D$. 
Figure 9.3 Coherent backscattering of light measured with two different values for the transport mean free path $\ell$. The typical angular width varies as $\lambda/\ell$. Narrow cone: a sample of BaSO$_4$ powder with $\ell/\lambda = 4$; broad cone: TiO$_2$ sample with $\ell/\lambda = 1$. The inset confirms the triangular cusp predicted by diffusion theory, and also shows that the maximum enhancement factor is lowered for the sample with a small value of $\ell/\lambda$. (Reprinted from Ref. 55.)

The term associated with coherent backscattering may be expressed as the integral

$$\langle I_{c} (\theta) \rangle = \int \int P(s, \rho, k) \cos (k \cdot \rho) ds d^2 \rho,$$  \hspace{1cm} (9.16)

where $P(s, \rho, k)$ is the probability density that a photon incident upon the medium will exit the sample at a position displaced by $\rho$ with wave-vector $k$ after following a path of length $s$ within the medium. Since the distribution of scattering angles for the last scattering event in the path is broad, we expect that $k$ will not be correlated with $s$ or $\rho$, which relate to the entire wave trajectory within the sample. We therefore express $P(s, \rho, k)$ as a product

$$P(s, \rho, k) = A(s, \rho) B(k),$$  \hspace{1cm} (9.17)

where $B(k)$ is the specific intensity for reflected radiation. Integrating $A(s, \rho)$ over $s$ gives $I_{0}(\rho)$, the point spread function, which is the transverse spatial distribution of intensity on the surface due to excitation of a point on the surface by a normally incident wave. Thus, the coherent backscattering peak is
the Fourier transform of the point spread function on the incident surface multiplied by $B(k)$

$$
\langle I_c(\theta) \rangle = B(k) \int I_0(\rho) \cos(k \cdot \rho) d^2 \rho. \tag{9.18}
$$

In weakly scattering media with $k \ell \gg 1$, the coherent backscattering peak is much narrower than $B(k)$. The width of $I_0(\rho)$ is on the order of a few mean free path lengths $\Delta \rho \approx \ell$, since most of the incident energy returns to the surface after only a few scattering events. Because components of $\rho$ and the corresponding components of $k$ along the same direction are conjugate variables in the Fourier transform relation between the coherent backscattering peak and the point spread function, we have

$$
\Delta k_{||} \sim k \Delta \theta \sim \frac{1}{\Delta \rho} \approx \frac{1}{\ell}, \text{ or } \Delta \theta \approx \frac{1}{k \ell}, \tag{9.19}
$$

where $\Delta \theta$ is the width of the coherent backscattering peak. Because some of the reflected radiation survives many scattering events within the sample, $I_0(\rho)$ has a broad tail that can produce a sharp structure in its Fourier transform. This results in a sharp triangular peak in $\langle I_c(\theta) \rangle$,\textsuperscript{54} as seen in Fig. 9.3.\textsuperscript{55} For strong scattering, corresponding to small values of $\ell/(\lambda / 2 \pi) = k \ell$, the coherent backscattering peak broadens and corresponds to a sizable fraction of the total energy. As a result, when $\ell \approx \lambda$, the Fourier transform of $P(\rho)$ goes negative as the reflected radiation falls below the incoherent background level at large angles. This has been observed in recent measurements.\textsuperscript{61}

### 9.3 Modes

Numerous approaches have been taken to treat waves in random media. The methods of wave interference can be used to calculate the average of transmission and reflection, fluctuations, and correlation of scattered waves, all of which can be carried out by calculating the Green’s function. This approach is particularly useful when the Green’s function can be expressed as a perturbation expansion.\textsuperscript{62} However, it is difficult to obtain analytic results for localized waves where no small parameter exists and all of the diagrams must be summed. An approach that has proven useful when scattering is weak is the phenomenological radiative transfer method. The radiative transfer equation is a Boltzmann equation for the specific intensity $I_v(r,t)$ describing the flow of radiation in a disordered medium at position $r$, and time $t$ in a direction $\hat{v}$.\textsuperscript{67,68} This method can be used to derive a diffusion equation for the specific intensity and to sort out the complexities of the wave interaction with the surface. Random matrix theory is
another powerful approach that gives important results for the statistics of localized waves in quasi-one-dimensional samples. This method treats the reflection or transmission matrices for propagating waveguide modes whose elements are assumed random except for certain constraints. In the rest of this chapter, we will emphasize the underlying electromagnetic modes of the random medium, which can be directly observed for localized waves. We will see that fundamental characteristics of wave propagation can be described in terms of the statistics of modes.

9.3.1 Quasimodes

The field inside an open random sample may be viewed as a superposition of quasimodes, each of which corresponds to a volume speckle pattern of the field. The spatial and spectral variation of the polarization component $j$ of the field for the $n^{th}$ quasimode is given by

$$A_{n,j}(r,\omega) = a_{n,j}(r)\frac{\Gamma_n/2}{\Gamma_n/2 + i(\omega - \omega_n)},$$

(9.20)

where $\omega_n$ and $\Gamma_n$ are the central frequency and linewidth, respectively. In contrast to modes of a closed system, quasimodes are eigenstates associated with open systems, in which energy may be absorbed within the sample or may leak out through the boundaries. The spatial field distribution of a quasimode $a_n(r)$, satisfies the Helmholtz equation

$$Ha_n(r) = \left(\omega_n - i\Gamma_n/2\right)^2 a_n(r),$$

(9.21)

where $H$ is the non-Hermitian Helmholtz operator with open appropriate boundary conditions. The imaginary part of the corresponding eigenfrequency characterizes the broadening of the line. If a single one of these quasimodes was excited by a pulse, the local field amplitudes would decay exponentially in a time $2/\Gamma_n$. For the sake of simplicity, we will often refer to quasimodes as modes, without risk of confusion. Although, in general, the eigenstates of a non-Hermitian operator do not form a complete basis, Leung et al. have demonstrated that when the refractive index of materials in a leaky system varies discontinuously and approaches a constant asymptotic value sufficiently rapidly, the quasimodes are complete and orthogonal so that an arbitrary state of the system can be expressed as a superposition of quasimodes. In these cases, the Green’s function for $\omega^2 - H$ can be expanded with the use of these mutually orthogonal quasimodes to give
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\[
G(r_1, r_2, \omega) = \sum_n \frac{a_n(r_1) a_n(r_2)}{\omega^2 - (\omega_n - i\Gamma_n/2)^2}.
\] (9.22)

This expression encapsulates the equivalence between treatments that emphasize wave interference of multiply scattered trajectories arriving at a point and an analysis that stresses the decomposition of the field into the quasimodes of the wave within the sample.

An alternative way approach modes of an open system is to consider the waves both inside and outside of the medium. In nondissipative systems, the eigenstates with constant flux for the system as a whole have real eigenvalues. Because these states are complete and orthorgonal,78 the Green’s functions can be expressed in terms of these constant flux modes. This has proved useful in the study of the spatial structure of modes in random lasers and nonlinear systems.79 In the remainder of this chapter, we will emphasize the quasimode description of the wave inside the random sample, with complex eigenvalues reflecting leakage from the system and absorption within the system.

9.3.2 Localized and extended modes

The nature of wave propagation in disordered samples reflects the spatial extent of the wave within the medium.24,27,28 When modes are spectrally isolated, the envelope of the field inside the sample is exponentially peaked.27,29,31 On the other hand, modes that overlap spectrally may extend throughout the sample. In diffusive samples, eigenstates are spread throughout the sample, while in samples in which waves are generally localized, the intensity distribution is singly or multiply peaked. The emergence of a number of intensity peaks within samples that are longer than the localization length was explained by Mott as the hybridization of localized modes due to exponentially small overlap of excitation in neighboring localized states.80–83 Apart from an overall modulation on the wavelength scale, the spatial intensity distribution of such Mott states exhibits a number of peaks along the length of the sample equal to the number of coupled modes.45,47,82–84 Overlapping modes are observed even in samples in which the wave is typically localized. When the sample supports a number of roughly equally spaced localization centers, which have been termed necklace states by Pendry, their contribution to transmission is particularly large.47 Necklace states play a large role in transport since their increased spatial extent facilitates the advance of the wave through the sample and the resulting increased transmission occurs over a relatively broad frequency range. Because these states are short lived, they are relatively weakly affected by absorption, which strongly attenuates localized states.45,85,86 Conversely, occasionally the number of overlapping modes in diffusive samples may be small, and the lifetimes of such modes will tend to be larger than is typical. In the presence of gain, lasing will occur first in such long-lived quasi-extended modes.87–91 An important issue discussed in reviews of lasing in diffusive sample in this volume92–94 is the extent
to which the presence of gain modifies the modes of the sample when modes overlap spectrally and spatially.

The intensity distribution within the interior of a multiply scattering sample is generally inaccessible in three-dimensional samples, but can be examined in one- and two-dimensional samples.\textsuperscript{45,95} The presence of both isolated and overlapping modes within the same frequency range has been observed in measurements of field spectra carried out with a vector network analyzer inside a single-mode waveguide containing randomly positioned dielectric elements. Many of the elements had a binary structure so that a pseudogap is created in which the density of states was particularly low\textsuperscript{34} in the frequency range of the band gap of the corresponding periodic structure of binary elements. The waveguide containing the sample was slotted and covered with a movable copper slab so that measurements could be made at any point along the sample length without introducing substantial leakage through the top of the waveguide. Measurements in Fig. 9.4 show that spectrally isolated lines have Lorentzian shapes with the same width at all points within the sample and are strongly peaked in space. On the other hand, when modes overlap spectrally, the line shape varies with position within the sample and the spatial intensity distribution is multiply peaked. Quasi-extended intensity distributions within a region in which waves are typically localized arise from the superposition modes with Lorentzian lines that are multipeaked in space.\textsuperscript{45}

One-dimensional localization has also been observed in optical measurements in single-mode optical fibers\textsuperscript{96} and in single-mode channels that guide light within photonic crystals.\textsuperscript{46} When the structure bracketing the channel

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure9_4.png}
\caption{Intensity spectra of quasimodes versus positions within a random single-mode waveguide for (a) isolated and (b) overlapping waves. In (a) the intensity of the wave is exponentially localized, whereas in (b) the intensity distribution is multiply peaked and hence quasi-extended. (Reprinted from Ref. 45.)}
\end{figure}
is periodic, the velocity of the wave propagating down the channel experiences a periodic modulation so that a forbidden band is created. When disorder is introduced into the lattice, quasimodes with spatially varying amplitude along the channel are created. Modes near the edge of the band gap are long lived and readily localized by disorder. An example of spectra of vertically scattered light versus frequency for light launched down a channel through a tapered optical fiber is shown in Fig. 9.5.⁴⁶ The inset shows the disordered sample of holes with random departure from circularity in silicon-on-insulator substrates at a hole filling fraction of $f \sim 0.30$. The structure is produced using electron-beam lithography and chlorine-based inductively coupled plasma reactive ion etching.

Narrow spectral lines are also observed near the band edge of cholesteric liquid crystals.⁹⁷ In such anisotropic liquid crystals, the average molecular orientation, known as the director, rotates with depth into the sample to create a sample with planar anisotropy with constant pitch. A photonic band gap is formed at a wavelength within the medium equal to the pitch for light with the same sense of circular polarization as the handedness of the helical structure. Low-threshold lasing is observed in dye-doped cholesteric liquid crystals in long-lived modes at the band edge. When even small disorder is present, the wavelengths of the lasing modes differ from values of modes for a periodic sample.⁹⁷

We have considered the localization of waves inside samples with reduced dimensionality. Localization can also be achieved in samples that are homogeneous along one direction and inhomogeneous in the transverse plane.⁹⁸ For light incident along the direction in which the sample is homogeneous, the

**Figure 9.5** Spectrum of wave transmitted to a region within a single-mode photonic crystal waveguide near the short wavelength edge of the first stop band at $\sim 1520$ nm. The channel surrounded by irregular holes is shown in the inset. (Reprinted from Ref. 46.)
velocity of the wave is constant and so the variation of the waveform along this
direction in samples of different lengths provides the time evolution in the
transverse directions. The wave vector gains a transverse component as a result
of scattering in the transverse direction arising from disorder. Since the
component of the wave vector in the transverse direction $k_\perp$ is small, the product
of the transverse component of the wave vector and the mean free path in the
transverse direction $k_\perp \ell$ is also small. This facilitates the observation of
localization. Linear and nonlinear localization have been observed in disordered
two-dimensional lattices written into photorefractive materials by superimposing
a random speckle pattern upon three interfering beams, which alone would
produce a hexagonal lattice.99 Exponential localization in the plane of the
disordered lattice is achieved even though the index modulation is small.
Transverse localization was also produced in a one-dimensional lattice of
coupled optical waveguides patterned on an AlGaAs substrate.100 Light is
injected into one or a few waveguides at the input, and tunnels coherently
between neighboring waveguides as it propagates along the waveguide axis.
Samples with different strengths of disorder are created by changing the width of
the distribution of random widths of the waveguides, which are periodic on
average. At short lengths, a ballistic component is observed in the transverse
direction moving away from the point in the transverse direction at which the
light was injected. At greater sample lengths, the ballistic component disappears,
while an exponential central peak grows. Nonlinear perturbations enhance
localization in states with uniform phase in the transverse direction and tend to
delocalize waves in which the phase is modulated by $\pi$ rad in neighboring
waveguides.

A qualitative understanding of the characteristics of modes in localized
samples can be obtained by considering the intensity distributions and associated
spectra for samples with one or two defects within a periodic background. This
produces exponentially peaked states within the band gap. Computations for
the cases of a single defect either at the center of the sample [Figs. 9.6(a) and (b)] or
displaced from the center [Figs. 9.6(c) and (d)], and a pair of defect states [Figs.
9.6(e) and (f)] placed symmetrically about the center of the sample are plotted in
Fig. 9.6. The structure within each sample and the corresponding spatial intensity
pattern and transmission spectrum are shown. The underlying periodic structure
is a sample of binary elements with indices of refraction of 1 and 2. A quarter-
wave defect with index of refraction 1 and a thickness of one half the period is
introduced. This places the transmission peak for the single mode at the center of
the band gap.

For a single defect, the intensity falls exponentially from the defect, apart
from a modulation on the scale of $\lambda/2$. The modulation is characteristic of a
standing wave since the components of the wave propagating in opposite
directions are nearly of the same intensity except near the sample boundaries.
The difference in flux in the forward and backward directions is constant and
Figure 9.6 Simulation of propagation in one-dimensional samples with defects in a periodic structure, in which the refractive index alternates between 1 and 2 in segments of length 100 nm and period of 200 nm. (a) Spectrum and (b) intensity distribution for sample with single defects at the center. (c) Spectrum and (d) intensity distribution for samples with single defects equally displaced to the left or right of the center. (e) Spectrum and (f) intensity distribution for sample with two defects symmetrically displaced from the center. The defect is an additional 100 nm with refractive index 1. The exponential decay length is the same as that for the evanescent wave excited at the same frequency within the band gap in a defect-free structure. The transmission coefficient depends only on the exponential decay length and the shortest distance from the defect to the boundary. When the defect is in the first half of the sample, the intensity rises from the input to the defect site and then falls exponentially to the output surface. On the other hand,
when the defect is in the second part of the sample, the incident wave couples to
the defect state through an evanescent wave. The intensity first falls
exponentially until the point at which its value matches the value of the intensity
of the mode that is exponentially peaked at the defect. When the defect is at the
center of the sample, the distances the exponential rise and fall are equal and the
transmission coefficient is unity. This gives the largest integrated intensity within
the sample, relative to the intensity value at the boundaries of the sample, and
results in a small leakage rate of energy from the sample—corresponding to a
long lifetime and narrow linewidth. The peak intensity within the sample, as well
as the lifetime of the state, fall exponentially with displacement of the defect
from the center of the sample, which is simply related to the displacement from
the nearest boundary.

When two defects are positioned symmetrically with respect to the center of
the sample, the intensity at the first defect rises to the same level as in the sample
with a single defect at this position; however, the transmission is higher because
the intensity rises exponentially as the second peak is approached. Because the
intensity at the second defect is the same as at the first, and the distance to the
output from the second defect equals the distance to the input of the first defect,
the transmission coefficient is unity. In the case of two defects, two hybridized
modes with slightly shifted central frequencies are produced as a result of
coupling between the modes. In the sample with symmetrically positioned
defects, both modes have the same intensity distribution and a transmission
coefficient of unity. Spectrally isolated localized modes in random samples are
similarly exponentially peaked, while spectrally overlapping modes are coupled
to form multiply peaked quasimodes. Since both the peak transmissions and the
linewidths of the overlapping modes are typically much larger than those of
isolated modes, the overall transmission is then dominated by these multiply
peaked modes. Propagation within the sample can also be understood as the
coupling of a wave through single or multiple wells separated by barriers in
which the intensity falls exponentially. Measurements of quasimode
transformation as the sample is continuously modified show an anticrossing
behavior in which the closest approach of quasimode frequencies is determined
by the strength of coupling between localization centers.

9.3.3 Statistical characterization of localization

A dimensionless parameter that characterizes the statistical spatial and
spectral properties of modes of the medium is the degree of level overlap. This is
the ratio of the level width to the level spacing \( \delta = \delta \nu \Delta \nu \). This ratio is the
inverse of the average finesse of the spectrum, and is closely related to the
Thouless number, which is a measure of the sensitivity of the mode frequency to
a change of boundary conditions. Here,

\[
\delta \nu = \frac{< \Gamma >}{2\pi}
\]  

(9.23)
is the average frequency width of modes. This corresponds to the width of the field correlation function with frequency shift. The level spacing is the inverse of the density of states of the sample as a whole, and may be written as

$$\Delta \nu = \frac{1}{n(\nu)AL}, \quad (9.24)$$

where $n(\nu)$ is the density of states per unit volume per unit frequency, and $AL$ is the sample volume given by the product of the area and thickness of the sample. $\delta$ is an indicator of localization, since when $\delta \nu < \Delta \nu$, modes generally do not overlap spectrally and are exponentially peaked within the sample. The linewidth is narrow since the leakage rate from the sample, which is proportional to the ratio of the energy density near the boundaries to the integrated energy within the sample, is small when the intensity is strongly peaked within the sample. On the other hand, when $\delta \nu > \Delta \nu$, modes overlap spectrally and the wave is extended throughout the sample. Thus, $\delta = 1$ corresponds to the localization threshold.

The critical dimension for localization can be found by considering the scaling of $\delta$. For diffusive samples $\delta \nu \sim D/L^2$, where $D$ is the diffusion coefficient controlling the spread of intensity or, correspondingly, the migration of particles in random systems and consistent with the diffusion relation

$$<|r(t) - r(0)|^2> = 6Dt. \quad (9.25)$$

The level spacing is inversely proportional to the volume $\Delta \nu \sim 1/L^d$ for samples with $A = L^{d-1}$. In this case, $\delta$ scales as $\delta \sim L^{d-2}$, and $\delta$ decreases with increasing $L$ for $d < 2$. On the other hand, $\delta$ increases with $L$ for $d > 2$ for waves that are diffusive on a scale not much larger than the mean free path. Thus, we find again that $d = 2$ is the critical dimension for localization.

The relationship between the degree of mode overlap $\delta$ and transmission can be seen by drawing an analogy between transmission of classical waves and electronic conductance. In the absence of inelastic processes, the suppression of average conductance due to the enhanced return of the wave to a coherence volume is connected to the average value of the dimensionless conductance $g = G/(e^2/\hbar)$. Landauer showed that the dimensionless conductance may be expressed in terms of scattering coefficients. When measured between points within perfect leads at a distance from the sample of the order of a few wavelengths so that all evanescent waves have decayed, this relation is

$$g = \langle T \rangle = \sum_{ab} \langle T_{ab} \rangle, \quad (9.26)$$

where the transmittance $T$ is the sum of transmission coefficients $T_{ab}$ over all of the $N$ input and outgoing transverse modes $a$ and $b$, respectively. Thus, $g$
Chapter 9 describes classical transmission as well as electronic conductance. Using the Einstein relation for the conductivity

\[ \sigma = \left( \frac{e^2}{h} \right) D n(\nu), \]  

(9.27)

the conductance can be written as

\[ G = \frac{\sigma A}{L} = \left( \frac{e^2}{h} \right) \frac{D n(\nu)A}{L}. \]  

(9.28)

We then find that

\[ g = \frac{G}{(e^2 / h)} = \frac{D n(\nu)A}{L} = \frac{\delta \nu}{\Delta \nu} = \frac{1}{\delta}. \]  

(9.29)

However, we have seen that the degree of mode overlap \( \delta \) is a measure of localization. Thus \( g \), which is a measure of average electronic transport, is also a measure of localization, which occurs at a threshold \( g = \delta = 1 \). \(^{28}\)

In quasi-one-dimensional samples, which have reflecting sides and are much longer than their transverse dimensions \( L \gg \sqrt{A} \), \( g = <T> \sim N<T_a> \), where \( N \sim 2\pi A/\lambda^2 \) for polarized radiation and

\[ T_a = \sum_b T_{ab} \]  

(9.30)

is the coefficient of total transmission for a phase coherent incident wave. This may be a single incident transverse mode \( a \), such as the plane wave or a propagating mode of a cavity, or a local coherent source placed near the sample. In the diffusive limit \( g \gg 1 \), \( \langle T_a \rangle \approx \ell / L \) and can be approximated by \( g \approx N\langle T_a \rangle \approx N\ell / L \). If the cross-sectional area \( A \) is fixed as the length increases, as is the case in a long wire or microwave waveguide, \( g \) will always fall below unity. Thus, waves will always be localized in sufficiently long quasi-one-dimensional static samples and electrons will always be localized in sufficiently long wires, when the temperature is low enough that dephasing can be neglected. \(^{27}\) For \( L \approx N\ell \), \( g \approx 1 \), suggesting that the localization length in quasi-one-dimensional samples is \( \xi \approx N\ell \).

The connection of intensity correlation to spatial localization can be seen by noting that the degree of correlation is related to the finesse \( l/\delta \). For monochromatic excitation of waves in a random medium in which the ensemble
average of transport is described by the diffusion equation, the excitation frequency falls within the half-width of \( \delta \) modes on average, since approximately \( \delta \nu / 2 \Delta \nu \) modes just below and a similar number just above this frequency overlap the excitation frequency. This relationship is shown schematically in Fig. 9.7(a). The field within the medium can be roughly represented by the superposition of these resonantly excited modes. Since the wave within the sample is approximately specified by the amplitudes of these \( \delta \) modes, it may be specified roughly by \( \delta \) parameters. As a result, the degree of correlation of intensity is approximately \( 1 / \delta \), or equivalently, \( 1 / \gamma \). For localized waves, the excitation frequency generally either resonantly excites a single mode or falls between modes, as seen in Fig. 9.7(b). The fractional correlation of intensity or total transmission is then dominated by the on-resonance contribution. Because transmission is appreciable only on resonance, the typical value of transmission on resonance as compared to the average value for all frequencies is

\[
\frac{\Delta \nu}{\delta \nu} = \frac{1}{\delta}.
\]  

(9.31)

Since the wave is only on resonance for a fraction \( \delta \nu / \Delta \nu = \delta \) of the entire spectrum for localized waves, and in this case,

\[
\frac{T_a}{\langle T_a \rangle} \gg 1, \quad \delta s_a = s_a - \langle s_a \rangle = \frac{T_a}{\langle T_a \rangle} - 1 \sim s_a,
\]  

(9.32)

we find that \( \text{var}(s_a = T_a / \langle T_a \rangle) = \langle (\delta s_a)^2 \rangle \sim (1/\delta)^2 \delta \sim 1/\delta \).

The degree of correlation at the output of the sample between two points for which the field correlation function vanishes, or between orthogonal propagation modes of the region outside the sample \( b \) and \( b' \), \( \gamma = \langle \delta s_{ab} \delta s_{ab'} \rangle \), can be shown to equal the variance of total transmission normalized to its ensemble average, \( \gamma = \text{var}(s_a = T_a / \langle T_a \rangle) \).\footnote{22}

The degree of correlation is also inversely proportional to the dimensionless conductance \( \kappa \sim 1/\gamma \). \footnote{6,22,43,72,73} We conclude that \( \kappa = \text{var}(s_a) \sim 1/\delta \) for localized as well as for diffusive waves. In the diffusive limit for nonabsorbing samples, \( \kappa = 2/3g = 2/3\delta \).

We have seen above that the parameters \( \delta \), \( \text{var}(s_a) \), \( \kappa \), and \( g \) are intimately connected. The presence of spectrally isolated modes for small values of \( \delta \) signals the spatial localization of modes of the sample and corresponds to large fluctuations of transmission from the average value. \( \text{Var}(s_a) \) is a measure of the size of these fluctuations. However, when the speckle pattern is normalized to the total transmission, the field is a Gaussian random variable. This has been demonstrated in measurements of field statistics\footnote{22} and is a fundamental
assumption of random matrix theory. Since the intensity for Gaussian waves is not correlated on scales greater than the wavelength, fluctuations in the speckle pattern of the normalized field self-average to give a contribution to \( \text{var}(s_a) \) of \( \sim 1/N \). The large fluctuations observed in transmission are rather due to the value of long-range intensity correlation \( \kappa \). Thus, the output field glows brightly or dimly as a whole as the frequency is tuned. Large fluctuations in transmission are associated with random ensembles with small values of \( g \) because the incident wave is then primarily off resonance with modes of the structure, and decays exponentially within the sample. Transmission may be large, however, when the incident wave is resonant with modes of the sample. Azbel showed that the transmission coefficient on resonance approaches unity when the wave is localized near the center of the sample, but falls exponentially with separation from the center of the sample. The four localization parameters discussed above are related in the absence of absorption as follows: \( \delta = g = 2/3\text{var}(s_a) = 2/3\kappa \). The equality \( \kappa = \text{var}(s_a) \) is maintained even in the face of absorption, while the equality \( \delta = g \) is not. In absorbing samples, \( \delta \) increases because linewidths are broadened by dissipation, while the density of states and its inverse, the average level spacing, does not change. However, \( g = \langle T \rangle \) decreases when absorption is present. Thus the ratio \( \delta g \) increases with absorption.

Figure 9.7 Schematic diagram of normalized spectra for (a) spectrally overlapping and (b) spectrally isolated modes.
In order to experimentally investigate the localization transition, it is useful to find a measurable localization parameter that scales exponentially with thickness and is not drastically affected by absorption. Localization parameters should exhibit a weak dependence on absorption, since the spatial distribution of quasimodes is hardly affected by moderate absorption. A parameter that measures localization should also reflect the statistical character of wave transport. We start by considering the distribution of normalized total transmission $P(s_a)$. An expression for $P(s_a)$ as a function of $g$ was found in the diffusive limit without absorption by Van Rossum and Nieuwenhuizen\cite{15,72} using diagrammatic calculations, and by Kogan and Kaveh\cite{16} using random matrix theory. The distribution $P(s_a)$ is found to be a function of a single parameter $g$ with $\text{var}(s_a) = 2/3g$, $P(s_a)$. It is therefore possible to express $P(s_a)$ as a function of $\text{var}(s_a)$. The functional form of the distribution for $x = s_a$ in terms of its variance $\beta = \text{var}(x)$, is\cite{15,72}

$$
\begin{align*}
P(x) &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \exp(qx) F\left(\frac{3q\beta}{2}\right) dq, \\
F(q) &= \exp \left[-\frac{2\ln^2(\sqrt{1+q^2}+\sqrt{q})}{3\beta}\right].
\end{align*}
$$

(9.33)

Though the direct link between $g = <T>$ and localization breaks down in the presence of absorption, so that $P(s_a)$ is no longer a function of $g$, $P(s_a)$ may still be expressed in terms of the variance of the distribution by Eq. (9.33).\cite{74}

Detailed statistical measurements of microwave transmission are carried out in ensembles of statistically equivalent random quasi-one-dimensional samples contained in a copper tube. The samples are generally dielectric spheres with diameter of ~ 1 cm, which is of the order of one-half of the tunable microwave wavelength. The mean free path $\ell$ is determined by the local scattering strength, which depends on the dielectric constant of the sphere $\varepsilon$, the ratio of the sphere diameter to the wavelength, and the sphere density, which can be varied by placing the spheres in Styrofoam shells. Frequently studied samples are Polystyrene with index of refraction $n = 1.59$, and alumina with $n = 3.14$, where $\varepsilon = n^2$. The dimensionless conductance $g = N\ell / L$ is determined by the local scattering strength and the sample cross-section and length, as well as the wavelength. Spectra of the polarized field transmission coefficient at points on the output are obtained by using a vector network analyzer from the ratio of the transmitted to the incident field detected with a short wire antenna. The amplitude and phase of the field are obtained from the measurement of the in-phase and out-of-phase components of the field. Field spectra in the frequency domain can be transformed to yield the temporal response to pulsed excitation. The source is a horn or wire antenna. Measurements can be taken at a single point or over a grid of points with spacing sufficiently tight that the two-dimensional sampling theorem\cite{109} can be used
to obtain the full speckle pattern. After each spectrum is taken, the sample is briefly rotated to create a new random configuration.

The intensity is obtained by squaring the field amplitude, while the total transmission can be obtained either from the sum of intensity in the transmitted pattern or more rapidly by using an integrating sphere. The integrating sphere is comprised of a shell with diameter much greater than the diameter of the sample tube, filled with movable scatterers and with a reflecting outer sphere. The integrating sphere is rotated so that the speckle pattern within the shell is scrambled. The time average intensity detected by a Schottky diode in the center of the shell, at a particular frequency is thus proportional to the total transmission.

Measurements of $P(s_a)$ for microwave radiation transmitted through samples of randomly mixed Polystyrene spheres contained in copper tubes of different length and diameter are shown in Fig. 9.8(a). These measurements are seen to be in agreement for strongly correlated absorbing samples with values of $\text{var}(s_a)$ much smaller than and much greater than unity for weak and strong absorption. Thus, Eq. (9.33) describes the statistics of transmission far from the regime in which it was derived. It is perhaps even more surprising, as will be seen below, that Eq. (9.33) also describes the statistics of the speckle pattern evolution when the incident frequency is changed.

Long-range correlation of intensity across the output face of quasi-one-dimensional samples reflects fluctuations of total transmission. The field in individual transmitted speckle patterns is a Gaussian random variable and is correlated on a short length scale of $\lambda/2$. The Gaussian nature of the field in single speckle patterns is a central assumption of random matrix theory. This assumption is confirmed in the probability distribution measurements of the field in random ensembles and in observations within individual speckle patterns. Because the intensity is the square of the field, the circular Gaussian distribution of the field

$$P(r,i) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left[-\frac{(r^2 + i^2)}{2\sigma^2}\right], \quad (9.34)$$

where $r$ and $i$ are the in-phase and out-of-phase components of a single polarization component of the field, respectively, leads to a negative exponential distribution for the normalized polarized intensity

$$s_{ab} = \frac{T_{ab}}{\langle T_{ab} \rangle} \frac{1}{s_a} \exp\left(-\frac{s_{ab}}{s_a}\right), \quad (9.35)$$

with

$$\text{var}\left(\frac{s_{ab}}{s_a}\right) = \langle \frac{s_{ab}}{s_a} \rangle = 1.^{1,2,4} \quad (9.36)$$
Wave Interference and Modes in Random Media

Figure 9.8 Probability distribution functions of (a) normalized total transmission $P(s_a)$ and (b) normalized transmitted intensity $P(s_{ab})$, respectively, for three Polystyrene samples with dimensions: a) $d = 7.5$ cm, $L = 66.7$ cm; b) $d = 5.0$ cm, $L = 50$ cm; c) $d = 5$ cm, $L = 200$ cm. Solid lines are given by Eqs. (9.33) and (9.37) using measured values of var($s_a$) of 0.50, 0.65, and 0.22 for samples a), b) and c), respectively. The dashed line in (b) is a plot of the Rayleigh distribution $P(s_{ab}) = \exp(-s_{ab})$. (Reprinted from Ref. 74.)

The distribution of transmitted intensity normalized by its ensemble average value $P(s_{ab})$ is therefore a mixture of a negative exponential distribution and $P(s_a)^{16}$

$$P(s_{ab}) = \int_0^\infty \frac{ds_a}{s_a} P(s_a) \exp\left(\frac{-s_{ab}}{s_a}\right). \quad (9.37)$$

Measurements of $P(s_{ab})$ for the same samples for which measurements of $P(s_a)$ are shown in Fig. 9.8(a) are shown in Fig. 9.8(b) to be in agreement with plots of Eq. (9.37).

Thus, the distributions of intensity and total transmission are fully described by var($s_a$). We next consider the scaling of var($s_a$) in an absorbing quasi-one-dimensional sample over a scale of lengths in which the wave makes a crossover from photon diffusion to localization. This sample is composed of alumina spheres with diameter 0.95 cm and index of refraction 3.14 embedded within Styrofoam shells to produce a sample with alumina volume fraction 0.068.\textsuperscript{104} The
low sphere density in this sample ensures that distinct sphere resonances are not washed out. The sample is contained within a copper tube with diameter 7.3 cm and plastic end pieces. Measurements are made on large numbers of configurations by briefly rotating the tube between measurements. The variance spectrum of the transmitted intensity normalized by its ensemble average \( \text{var}(s_{ab}) \), for a 7.3-cm-diameter copper tube of length \( L = 80 \), shows a window of localization between 9.9 and 11 GHz (see Fig. 9.9). The window is considerably narrower in shorter samples. As a consequence of Eq. (9.37), the moments of \( s_a \) and \( s_{ab} \) are related by

\[
\langle s_{ab}^n \rangle = n! \langle s_a^n \rangle .^{16}
\] (9.38)

This gives

\[
\text{var}(s_{ab}) = 2 \text{var}(s_a) + 1 ,
\] (9.39)

so that the localization threshold given by \( \text{var}(s_a) = 2/3 \) occurs at \( \text{var}(s_{ab}) = 7/3 \). \( \text{Var}(s_{ab}) \) rises above the localization threshold just above the first Mie resonance of the spheres with \( n = 3.14 \), which is located around frequency \( \nu \approx 2\pi a/c \), where \( a \) is the radius of the sphere. The average transit time is peaked, while the transport velocity exhibits a dip on resonance. Localization is not achieved on resonance in this sample even though \( \delta \nu \) is small as a consequence of the lengthened transit time in the sample because the density of states is also peaked on resonance. This leads to a dip in the level spacing and a level overlap parameter exceeding unity, \( \delta = \delta \nu / \Delta \nu > 1 \). The strongest scattering in this medium is achieved at 10.0–10.2 GHz. In this range, \( k\ell \sim 2 \). Somewhat higher values of \( k\ell \) are reached at greater sphere density.

The scaling of \( \text{var}(s_a) \) in this sample attests to its utility as a localization parameter. This quantity scales linearly for values of \( \text{var}(s_a) < 2/3 \), but scales exponentially for larger values. Measurements of the scaling of \( \text{var}(s_a) \) are shown in Fig. 9.10.

**9.3.4 Time domain**

The statistics of fluctuations and correlation can be studied in the time domain as well as in the frequency domain. The linewidth \( \Gamma_n \) represents the decay rate of the mode in the time domain due to both leakage in an open system and absorption. The statistics of \( \Gamma_n \) and of the delay time have been extensively studied based on the statistics of the scattering matrix, \( S \), since the complex eigenfrequencies of the quasimodes \( \omega_n - \frac{i}{2} \Gamma_n \) correspond to the poles of the \( S \)-matrix, and the Wigner delay time
Figure 9.9 Spectrum of var($s_{ab}$) in sample with length $L = 80$ cm. The dashed line indicates the localization threshold. A window of localization in which var($s_{ab}$) > 7/3, corresponding to var($s_a$) > 2/3, is found just above the first Mie resonance of the alumina spheres comprising the sample. (Reprinted from Ref. 110.)

Figure 9.10 Scaling of var($s_a$) in samples of alumina spheres. Above a value of the order of unity, var($s_a$) increases exponentially. (Reprinted with permission from Ref. 43.)
\[
\tau_w (\omega) = \frac{1}{\omega} \text{Tr} \left( -i \hat{S}^t dS / d\omega \right) 
\]  
(9.40)
effectively characterizes the energy decay rate. For a review on this issue, see Ref. 112 and references therein.

Experimental investigation of the statistics of dynamics can take place directly via measurements following pulsed excitation or by Fourier transforming the product of transmitted field spectra and the spectrum of the incident pulse.\textsuperscript{18,113–123} Measurements in the time domain make it possible to unravel the effects of absorption. Absorption in homogeneously absorbing samples simply introduces a multiplicative exponential factor in time.\textsuperscript{40,43} All paths emerging from the sample at a given delay time \( t \) have the same length and have suffered the same diminution due to absorption. Because the weight of the path distribution is not changed by absorption at a given \( t \), and since all partial waves arriving at a fixed time are equally suppressed by absorption, weak localization is not affected by absorption in the time domain.\textsuperscript{40,43,117,122} The influence of localization can be seen in a reduction of the decay rate with increasing time delay,\textsuperscript{117,122} and in the increase of \( \kappa = \text{var}(s_a) \) with time.\textsuperscript{119}

In pulsed microwave measurements through diffusive samples, transmission is expected to fall exponentially at a rate equal to that of the lowest diffusion mode

\[
1/\tau_i = \pi^2 D / (L + 2\zeta_0)^2 
\]  
(9.41)
after a time \( \tau_i \) in which higher order modes with decay rates

\[
1/\tau_n = n^2 \pi^2 D / (L + 2\zeta_0)^2 
\]  
(9.42)
have largely decayed. A breakdown of the diffusion model was found in quasi-one-dimensional random dielectric media composed of random mixtures of low-density alumina spheres at frequencies far from the localization window for which \( \kappa = 0.09, 0.13, 0.25, \) and 0.125, respectively, in samples A–D.\textsuperscript{117} The decay rate for the transmitted pulses shown in Fig. 9.11(a) for different random ensembles is seen in Fig. 9.11(b) to fall in time at a nearly constant rate. A linear decay of the decay rate would be associated with a Gaussian distribution of decay rates for quasimodes of the medium.\textsuperscript{124} A slightly more rapid decrease of the decay rate is associated with a slower than Gaussian fall-off of the distribution of mode decay rates. The pulsed transmission can be regarded as a distribution of decaying modes. The slowing down of the decay rate at long times reflects the survival of more slowly decaying modes at long times. The decay rate distribution can then be found by taking the Laplace transform of the transmitted pulse intensity.\textsuperscript{117,124}
Figure 9.11 (a) Average pulsed transmitted intensity in alumina samples of \( L = 61 \) cm (A), 90 cm (B and D), and 183 cm (C). Sample D is the same as sample B except for the increased absorption by titanium foil inserted along the length of the sample tube D. (b) Temporal derivative of the intensity logarithm gives the rate \( \gamma \) of the intensity decay due to both leakage out of the sample and absorption.

The temporal variation of transmission can also be described in terms of the growing impact of weak localization on the dynamic behavior of waves, which can be expressed via the renormalization of a time-dependent diffusion constant or mean free path.\(^{120}\) The decreasing decay rate has also been explained using a self-consistent theory of localization,\(^{125-129}\) which incorporates an effective diffusion coefficient that varies with depth inside the sample\(^{126-128}\) and with the frequency of modulation of the incident intensity. Though the sample is diffusive, significant suppression of the leakage rate is observed. At the same time, the extent of intensity fluctuations and correlation increases with time delay.\(^{119}\) However, the field correlation function with displacement and polarization rotation is the same for steady state and pulsed transmission. This reflects the Gaussian statistics within the speckle pattern of a given sample configuration. The intensity correlation function at any time delay has the same form as in steady state, but exhibits a time-varying degree of correlation \( \kappa_c(t)\)\(^{119}\) that depends on the spectral bandwidth of the pulse \( \sigma \). The total transmission and intensity distributions at various delay times are given using Eqs. (9.36) and (9.37) with the value of

\[
\alpha(t) = \text{var}(s_c)(t) = \kappa_c(t). \tag{9.43}
\]
The increasing value of $\kappa_c(t)$ reflects the sharpened spectrum of transmission at the given time delay. This reflects the greater prominence of longer-lived modes that have narrower linewidths.

Microwave measurements of pulsed transmission at frequencies within the localization window show an evolution of the nature of propagation with increasing time delay reflecting the reduced role of short-lived overlapping states and the growing weight of long-lived spectrally isolated modes with increasing delay from the incident pulse. At short times, propagation is diffusive; at intermediate times, transport can be described in terms of a position and frequency-dependent renormalized diffusion coefficient; while at later times, energy flows largely from isolated localized modes and can be described via a distribution of localized modes in accord with the single parameter scaling theory of localization. Different pulse dynamics for modes with different degrees of spatial and spectral overlap can be directly observed in measurements in single-mode waveguides. Results are consistent with a change from localization to diffusion-like dynamics with an increasing degree of overlap in particular configurations. The distinction between isolated and overlapping modes is also seen in pulsed optical spectra in random porous silicon slabs. A full theory of dynamics could be obtained from the statistics of mode spacing and overlap as a function of $\delta$.

Störzer et al. also recently observed a falling decay rate in optical diffusion through a slab of particles of TiO$_2$ in the rutile phase with a refractive index of 2.8. The mean free path is determined from measurements of the coherent backscattering peak, while the pulse transmission profile is obtained from a histogram of time delays for single photons traversing the slab relative to the incident pulse. Substantial reductions in transmission relative to that predicted by a diffusion model are observed. The results for $k\ell$ with values of 4.3 and 2.5 are shown in Fig. 9.12.

### 9.3.5 Speckle

We have seen that fluctuations, correlation, and localization are closely related in quasi-one-dimensional samples. These characteristics of propagation can be assessed only in the context of an ensemble of random samples. Indeed, the statistics within an individual speckle pattern are independent of the nature of transport. For an ensemble of speckle patterns, the probability distribution of the field is Gaussian, once their average intensities are normalized. Correspondingly, only short-range intensity correlation is observed and the cumulant intensity correlation function equals the square of the field correlation function. Typical speckle patterns for diffusive and localized waves are shown in Figs. 9.13(a) and (b). It is seen that the spatial variation of intensity is correlated with the variation in phase. Near an intensity peak, phase change is small, while in low-intensity regions, the phase change is rapid. Most importantly, phase
singularities,\textsuperscript{133–144} at which all equiphase lines cross, appear at points of vanishing intensity where the phase is not defined. In a Gaussian random wave field, phase singularities are generic and the density of phase singularity is about twice that of intensity maxima.\textsuperscript{3,139–141} Although the sizes of speckle spots differ in the two patterns due to the different wavelengths, the statistics of the structure of intensity and phase distributions for the normalized patterns for an ensemble of random samples are essentially the same. This is reflected in the statistics of the field near the core of phase singularities. The phase and intensity variation near a phase singularity has a simple geometric structure,\textsuperscript{141,142,144} which is illustrated in Fig. 9.14 (a). The contours of constant intensity and constant current magnitude are ellipses and circles, respectively. The orientation and eccentricity $\varepsilon$ of the ellipses determine the phase variations in the vortex core.\textsuperscript{144} The magnitude of the current increases linearly with the distance from the core, with a
Figure 9.13 Examples of speckle patterns at the output of a quasi-one-dimensional sample for (a) diffusive and (b) localized waves. The gray scale shows the intensity variation and the colored lines are equiphase lines. Green dots represent phase singularities. (Reprinted from Ref. 23.)

Figure 9.14 (a) Core structure of a phase singularity. The straight lines are equiphase lines with phase values shown in Fig. 9.10. Circles (green) are current contours, while ellipses (white) are intensity contours. (b) Probability distribution of $\varepsilon$. (c) Probability distribution of $\Omega'$. The solid line is a derivation from Gaussian statistics of random fields. Green triangles and red circles are experimental data for diffusive and localized waves, respectively. (Reprinted from Ref. 144.)
slope known as the vorticity, and denoted as $\Omega$. Since the average intensity across the entire speckle pattern is proportional to the total transmission $s_a$, we define a normalized vorticity

$$\Omega' = \Omega / s_a, \quad (9.44)$$

to examine the relative intensity variation within one speckle pattern. The probability distributions of $\varepsilon$ and

$$\tilde{\Omega}' = \Omega' / \langle \Omega' \rangle, \quad (9.45)$$

shown in Figs. 9.14 (b) and (c), demonstrate that statistical distributions of phase and intensity in speckle patterns for diffusive and localized waves are the same. Both $P(\varepsilon)$ and $P(\tilde{\Omega}')$ can be derived from the Gaussian statistics of the random field. The variation of vorticity with the $\text{var}(s_a)$ is linear

$$\text{var}(\tilde{\Omega}) = \left[ 3 \text{var}(s_a) + 1 \right] / 2, \quad (9.46)$$

so that $\text{var}(\tilde{\Omega})$ is a direct measure of localization.

Speckle patterns depend on the spatial configurations of all scatters in the sample and thus provide a fingerprint of the sample. Transmission for a given sample configuration can be fully characterized by the electromagnetic modes of the medium. Each mode is associated with a distinctive speckle pattern. In an open or dissipative sample, each mode has a finite linewidth. When $\delta$ is not too large, the modes that are superposed to produce the speckle pattern at a specific frequency can be determined. This superposition of fields can be expressed as

$$E_j(x, y, \omega) = \sum_n a_{n,j}(x, y) \frac{\Gamma_n / 2}{\Gamma_n / 2 + i(\omega - \omega_n)}, \quad (9.47)$$

where $j$ is the polarization index and $a_{n,j}(x, y)$ is the spatial distribution of the field for the $n$th mode. When several modes overlap, the speckle pattern varies with frequency because the complex amplitudes of different modes are determined by a frequency-dependent Lorentzian factor. For diffusive waves, the incident wave always falls within the linewidth of several electromagnetic modes. Speckle patterns therefore change relatively uniformly with frequency since the amplitudes of a large number of modes contributing to the speckle pattern change. In contrast, when the wave is localized, the modes do not overlap and transmission increases dramatically as the source is tuned on resonance. When the incident frequency is on resonance with a single mode, transmission is
dominated by the interaction with that mode. The speckle pattern then essentially corresponds to the distinctive pattern of the resonantly excited mode. Though the magnitude of transmission will vary appreciably as the frequency of the wave is tuned through resonance, the change in the spatial distribution of the field is small since the pattern is determined by the interaction with a single mode. However, once the frequency is tuned sufficiently far off resonance, the spatial distribution of the wave will change since the contribution of the initially resonant mode may be reduced to the level of the contributions from other modes. So, in contrast to the continual variation of the speckle pattern with frequency shift in diffusive case, the speckle pattern may change abruptly as the source frequency is tuned for localized waves. Therefore, the statistics of speckle evolution with frequency shift reflects the degree of overlap of modes and can thus distinguish between diffusive and localized waves.

The speckle pattern changes when the internal structure of the sample changes, as may occur in colloidal samples\textsuperscript{145–146} or when the frequency is tuned.\textsuperscript{99} The difference in the evolution of speckle patterns for diffusive and localized waves may be quantified by following any of a number of the features within the speckle pattern or quantities computed for the speckle pattern as a whole. Below, we define three quantities that characterize the evolution of the speckle pattern. First, since the speckle pattern can be sketched by a number of critical points, e.g., maxima, minima, and saddle points of intensity or phase, the change in speckle patterns are roughly proportional to the displacement of these points. Here, we consider only the phase singularities and define $\tilde{R}$ to be the average displacement of phase singularities. We also consider measures of the overall changes of phase and intensity: $\tilde{\sigma}_{\Delta \phi}$, the standard deviation of phase changes

$$\Delta j(x, y) = j(x, y; \omega + \Delta \omega) - j(x, y; \omega),$$ \hspace{1cm} (9.48)

and $\tilde{\sigma}_{I'}$, the standard deviation of fractional intensity change

$$\Delta I'(x, y) = \frac{2[I(x, y; \omega + \Delta \omega) - I(x, y; \omega)]}{I(x, y; \omega + \Delta \omega) + I(x, y; \omega)}.$$ \hspace{1cm} (9.49)

The tilde on these variables indicates that the variables are normalized by their ensemble averages. All these quantities characterize a relative change of speckle pattern but do not characterize the absolute changes of the field. When

$$E_j(x, y; \omega + \Delta \omega) = c \cdot E_j(x, y; \omega),$$ \hspace{1cm} (9.50)
where $c$ is a complex constant, the change of phase $\Delta \phi$ and intensity $\Delta I'$ are both uniform over the full pattern and then,

$$\tilde{R} = \tilde{\sigma}_{\Delta \phi} = \tilde{\sigma}_{\Delta I'} = 0,$$

(9.51)

and the speckle pattern is unchanged. This occurs when the incident frequency is near resonance with a single mode. We find that for a small change in the speckle pattern, these three quantities are proportional on average. A linear relation between the average value of $\tilde{R}$ for given $\tilde{\sigma}_{\Delta \phi}$ (or $\tilde{\sigma}_{\Delta I'}$) and $\tilde{\sigma}_{\Delta \phi}$ (or $\tilde{\sigma}_{\Delta I'}$) is shown in Fig. 9.15.

The statistical properties of speckle evolution can be characterized by the probability distributions of $\tilde{R}$, $\tilde{\sigma}_{\Delta \phi}$, and $\tilde{\sigma}_{\Delta I'}$, which are shown in Fig. 9.16.23 Fluctuations in the change of speckle patterns are much larger for localized waves than for diffusive waves. Such enhanced fluctuation of $\tilde{R}$, $\tilde{\sigma}_{\Delta \phi}$, and $\tilde{\sigma}_{\Delta I'}$ for localized waves can also be seen in their spectra in a single configuration (Fig. 9.17). The spectrum of changes in the speckle pattern is continuous for diffusive waves and abrupt for localized waves. A comparison with spectra of $s_a$ in Fig. 9.15(b) further shows that the change of speckle pattern is small on resonance and greatly enhanced off resonance. Both Figs. 9.16 and 9.17 show that fluctuation of speckle changes is greatly enhanced in the localization transition. The distributions $P(\tilde{R})$, $P(\tilde{\sigma}_{\Delta \phi})$, and $P(\tilde{\sigma}_{\Delta I'})$ may all be described in terms of the corresponding variances utilized by Eq. (9.33), which described $P(s_a)$. The fit of Eq. (9.33) to the data is shown as the solid curves in Fig. 9.16.

The reason for the similarity between the statistics of changes of the speckle pattern and total transmission can be appreciated by comparing first- and second-order statistics of corresponding local quantities, such as the velocity of a single phase singularity and the intensity at a point.147 The degree of long-range correlation of the velocity of phase singularities is nearly equal to $\text{var}(\tilde{\sigma}_{\Delta I'})$; a similar relation exists between the degree of long-range intensity correlation $\kappa$ and $\text{var}(s_a)$. Once the velocity is normalized by $\tilde{\sigma}_{\Delta I'}$, the probability distribution and spatial correlation function of singularity velocity approaches that for Gaussian random wave fields. Similarly, when the intensity is normalized by the average intensity in the speckle pattern, the probability distribution of intensity and its spatial correlation function also become coincident with results for Gaussian random waves.

We have so far considered the statistics of locally three-dimensional samples within a quasi-one-dimensional geometry, which have a fixed cross-section and length much greater than the transverse dimensions. An important unknown is the statistics in the three-dimensional slab geometry. In this geometry, there is no well-defined area over which the total transmission $s_a$ can be defined.
Figure 9.15 Average of (a) $\tilde{R}$ versus $\tilde{\sigma}_{\Delta \varphi}$ for given $\tilde{\sigma}_{\Delta \varphi}$, and of (b) $\tilde{R}$ versus $\tilde{\sigma}_{\Delta I'}$ for given $\tilde{\sigma}_{\Delta I'}$. Results are plotted in triangles for diffusive waves and in circles for localized waves. (Reprinted from Ref. 23.)

Nonetheless, the intensity distribution given by Eqs. (9.33) and (9.37) coincided with that for transmitted ultrasound radiation in a slab of randomly positioned aluminum beads sintered to form an elastic network\textsuperscript{148} and for light transmitted through thick glass stacks.\textsuperscript{149} In the ultrasound measurements, the distribution corresponded to the quasi-one-dimensional distribution just beyond the localization threshold in one frequency range and otherwise to the distribution for diffusive waves. In thin glass stacks, in which light does not spread beyond a single coherence area, the intensity probability distribution at the output plane $P(s_{ab})$ is in accord with one-dimensional simulations, and falls sharply to zero as $s_{ab} \to 0$. As the sample thickness increases, the spatial intensity pattern undergoes a topological transformation in which phase singularities are
formed at intensity nulls as the lateral spread of the wave exceeds the field correlation length. $P(s_{ab})$ changes progressively to the distribution given in Eqs. (9.33) and (9.37). This distribution is a mixture of a mesoscopic distribution of $s_{a}$, $P(s_{a})$, and a distribution of intensity for Gaussian waves, even though there is no area for which the distribution of integrated intensity is given by the quasi-one-dimensional expression for $P(s_{a})$. These measurements show that mesoscopic intensity statistics have a universal form beyond one dimension and provide a measure of wave localization within the sample.
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Figure 9.17 Spectra of $s_a$ (black lines), $\tilde{R}$, $\tilde{\sigma_{\Delta \phi}}$, and $\tilde{\sigma_{\Delta I}}$, for (a) diffusive and (b) localized waves.

9.4 Conclusion

The striking similarities between the statistics of different quantities computed over the entire speckle pattern arise from the common excitation characteristics of the underlying electromagnetic modes of the medium. The statistics of modes depends on the degree of spectral overlap of modes. This may be expressed directly via the ratio of the width to the spacing of levels, and indirectly, but no less precisely, in terms of its effect on the variance of normalized total transmission $\text{var}(s_a)$, which equals the degree of intensity correlation $\kappa$, and is essentially equal to the inverse of the dimensionless conductance $g$.

Wave propagation may also be seen as a consequence of the crossing of partial waves within the sample. The interference of partial waves produces a volume speckle pattern. The return of partial waves to a coherence volume leads to weak localization and to the suppression of transport. Local fluctuations produced by the interference of partial waves propagate throughout the sample.
This leads to intensity correlation and enhanced fluctuations of intensity, total transmission, and conductance. These phenomena occur only when the wave is temporally coherent throughout the sample. They are referred to as mesoscopic because in the electronic context, they are observed only in samples at low temperatures and in samples that are intermediate in size between the atomic or microscopic scale and the macroscopic scale. For classical waves, scattering elements within a sample are generally correlated over scales much larger than the atomic size, so that the sample is essentially static and such coherence phenomena are observed at room temperature. All of the above interference phenomena may be readily interpreted in terms of modes. The speckle pattern is the superposition of field patterns for the modes of the medium, while the probability of return to a coherence volume, which is directly related to localization and mesoscopic fluctuations, is essentially equal to $1/\delta$. This may be seen by noting that $\delta = \delta/\Delta \nu$ may be expressed as the ratio of the Heisenberg time $\tau_H = 1/\Delta \nu$, which is the minimum time required for the wave to visit each coherence volume of the sample, and the Thouless time $\tau_{Th} = 1/\delta \nu$, which is proportional to the dwell time in the sample. Thus, $\tau_{Th}/\tau_H = 1/\delta$ is the average number of times the wave returns to a coherence volume. This demonstrates the close connection between the wave interference and mode pictures, each of which provides valuable insights into the nature of wave propagation and localization.

**Acknowledgments**

We thank Bing Hu, Patrick Sebbah, Andrey Chabanov, Jing Wang, Zhao-Qing Zhang, Jerome Klosner, and Shaolin Liao for valued interactions. This work was supported by the NSF under Grant No. DMR-0907285.

**References**


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