QUEENS COLLEGE
Department of Mathematics
Final Examination
$2\frac{1}{2}$ Hours

Mathematics 143 — Fall 2017

Instructions. Answer each question. Show your work and justify your answers. Partial credit will be awarded for relevant work.

1. Find each of the following integrals. Assume that k is a positive constant.

a. $\displaystyle\int \sin^4 x \cos^3 x \, dx$ b. $\displaystyle\int \frac{20}{(x+3)(x^2+1)} \, dx$ c. $\displaystyle\int \frac{x^2}{\sqrt{k^2 - x^2}} \, dx$

2a. Find the integral $\displaystyle\int \frac{\ln x}{x^2} \, dx$.

b. Decide whether the improper integral $\displaystyle\int_e^\infty \frac{\ln x}{x^2} \, dx$ is convergent or divergent. If convergent, evaluate without the use of a calculator.

3a. For $n \geq 1$, derive a reduction formula for the definite integral $I_n = \displaystyle\int_1^e (\ln x)^n \, dx$ in terms of $I_{n-1} = \displaystyle\int_1^e (\ln x)^{n-1} \, dx$ by using integration by parts.

b. Given that $I_1 = 1$, evaluate $I_2 = \displaystyle\int_1^e (\ln x)^2 \, dx$ in terms of e. Do not use the calculator.

c. Check your answer to part b by evaluating the integral on your calculator. (Include the calculator syntax that you used.)

4. Let $\tan^{-1}(x)$ be the inverse tangent function, also denoted $\arctan(x)$.

a. Write the geometric series expansion of $\frac{1}{1+x^2}$. What is the interval of convergence?

b. Find the Taylor series expansion for $\tan^{-1}(x)$, centered at 0 (also known as the Maclaurin series). *Suggestion.* Use $\tan^{-1}(x) = \displaystyle\int \frac{1}{1+x^2} \, dx$.

c. Calculate (to 10 decimal places) the partial sum s_{20} of 20 terms of the Taylor series expansion for $\tan^{-1}(x)$ in part b when $x = 1/\sqrt{3}$. (Include the calculator syntax you used.)

d. Find the numerical value of $6 \cdot s_{20}$ and explain whether your answer is reasonable.

5a. Decide whether the sequence $\left\{(n+1)^{1/n}\right\}$ is convergent or divergent. Justify your answer. If convergent, please evaluate the limit.

b. Decide whether the series $\displaystyle\sum_{n=1}^{\infty} (n+1)^{1/n}$ is convergent or divergent. Justify your answer.

(continued on other side)

6. Let $S_n = \sum_{k=0}^{n} \frac{4\,(-1)^k}{2k+1} = \frac{4}{1} - \frac{4}{3} + \cdots + (-1)^n \frac{4}{2n+1}$.

a. Evaluate S_{150} and S_{151} to 2 decimal places. (Include the calculator syntax you used.)

b. Prove that the series $\sum_{k=0}^{\infty} \frac{4\,(-1)^k}{2k+1}$ is convergent.

c. Explain why the sum of the series in part b lies between S_{150} and S_{151}.

d. Is the series in part b absolutely convergent? Justify your answer.

7. Let $f(x) = \sum_{n=0}^{\infty} \frac{n^2+1}{5^n}\,(x+2)^n$

a. Find the center and the radius of convergence of the power series for $f(x)$ above.

b. Find a power series for the derivative $f'(x)$ with the same center as in part a. What is the radius of convergence of the power series for $f'(x)$?

8. Let $T_n(x)$ be the Taylor polynomial of degree n with center at 0 for $F(x) = \sqrt{4-x}$ and let $R_n(x) = F(x) - T_n(x)$.

a. Write a general remainder formula for $R_n(x)$ according to Taylor's theorem.

b. Find $T_2(x)$.

c. Find a *numerical* bound for $|F(x) - T_2(x)|$, valid for all x such that $0 \leq x \leq 2$.

d. Find the first 3 *non-zero* terms in the Taylor series for $G(x) = \sqrt{4-3x^3}$, center at 0.