Department of Mathematics, Queens College
Math 152 Final Exam, Fall 2015

- **The exam has two parts.**
- **You have 150 minutes to answer the questions.**
- **Show all work in your exam book.**

Part I [30 points]. Clearly write the letter of the correct answer in your exam book. Show your work.

1. If $f(x) = 2x + \cos x$ and f^{-1} denotes the inverse of f, the derivative $(f^{-1})'(1)$ is

(A) $\dfrac{1}{2 - \sin 1}$ (B) $\dfrac{1}{2 + \sin 1}$ (C) $\dfrac{1}{2}$ (D) 2

2. $\displaystyle\lim_{x \to 0} \frac{\sin(e^{-x} - 1)}{x}$

(A) is -1 (B) is 1 (C) is 0 (D) does not exist

3. The improper integral $\displaystyle\int_2^\infty \frac{dx}{x \ln x}$ is

(A) $\ln 2$ (B) $\ln(\ln 2)$ (C) 2 (D) divergent

4. I set up an integral for the arc-length of $y = \ln x$ between $x = 1$ and $x = 3$. My calculator estimates this integral to be

(A) 2.3365 (B) 2.3289 (C) 2.3174 (D) 2.3019

5. The interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{4^n}{n}(x+1)^n$ is

(A) $\left(-\frac{5}{4}, -\frac{3}{4}\right)$ (B) $\left(-\frac{5}{4}, -\frac{3}{4}\right]$ (C) $\left[-\frac{5}{4}, -\frac{3}{4}\right)$ (D) $\left[-\frac{5}{4}, -\frac{3}{4}\right]$

6. The Maclaurin series of $f(x) = x\sin(2x^2)$ begins as

(A) $2x^3 + \frac{4}{3}x^7 + \frac{4}{15}x^{11} + \cdots$ (B) $2x^3 - \frac{4}{3}x^7 + \frac{4}{15}x^{11} - \cdots$

(C) $2x^3 + \frac{1}{6}x^7 + \frac{1}{120}x^{11} + \cdots$ (D) $2x^3 - \frac{1}{6}x^7 + \frac{1}{120}x^{11} - \cdots$

continued on the other side ⟶

Part II [70 points]. Solve the following six problems.

Problem 1. [12 points] In each case, find the derivative $y' = \dfrac{dy}{dx}$:

(a) $y = \tan^{-1}(x\, e^x)$

(b) $y = \sqrt[3]{\ln(x^4 + x)}$

(c) $y = (1 - 3x)^{\sin x}$

Problem 2. [18 points] Evaluate the following integrals:

(a) $\displaystyle\int \frac{\ln x}{x^3}\, dx$

(b) $\displaystyle\int \sqrt{25 - x^2}\, dx$

(c) $\displaystyle\int \frac{3x^3 + 4x - 6}{x^2(x^2 + 2)}\, dx$

Problem 3. [10 points] Let R be the region in the plane bounded by the curve $y = e^x$ and the lines $y = x + 1$ and $x = 1$.

(a) Sketch R and find its area.

(b) Use cylindrical shells to find the volume of the solid obtained by rotating R around the y-axis.

Problem 4. [8 points] Find the general solution of the differential equation

$$\frac{dy}{dx} = x \cos^2 y.$$

Then find the solution that satisfies the condition $y(0) = \pi/4$.

Problem 5. [12 points] Determine, using appropriate tests, the convergence or divergence of the following series:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2 + 1}$

(b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{7n^2 - 1}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n\, n!}{n^n}$

Problem 6. [10 points] Using your knowledge of the Maclaurin series of the exponential function, find the Maclaurin series of $f(x) = e^{-x^3}$ and use it to represent the integral

$$I = \int_0^1 e^{-x^3}\, dx$$

as an alternating series. Use this series to estimate the value of I with an error of less than 10^{-4}.