MATH 120 — Final Exam — 20 Dec 2022

Name: ______________________________

Class Section: __________________________

Page 1	
Page 2	
Page 3	
Page 4	
Page 5	
Total Score	
Grade	

Important Information:

Wait until you are told to start.

For the counting questions, leave your answers in terms of $n!$, $P(n,k)$, $\binom{n}{k}$, and $\left(\!\binom{n}{k}\!\right)$.

You are expected to SHOW YOUR WORK in all answers.
Wrong answers provided without work will receive no credit.

You may use the back of pages if you need more space or for scrap work.

If you continue work in a different location and want it to be graded,
indicate CLEARLY where it is.

1. Define the sets $A = \{1, 3, 5, 7, 9, \ldots, 101\}$ and $B = \{0, 11, 22, 33, 44, \ldots, 1100\}$.

   (a) Write A in set-builder notation.

   (b) Write the set $A \cap B$ in roster notation.

   (c) Give three elements of $A \times B$.

2. Determine whether each of the following sets is finite, countably infinite, or uncountably infinite. If the set is finite, determine its cardinality.

   (a) $\{q \in \mathbb{Q} : 0 \leq q \leq 0\}$

   (b) $\{25, 36, 49, 64, 81, 100, \ldots, 10000\}$

   (c) $\{45, 44, 43, 42, 41, 40, 39, 38, \ldots\}$

Page Score

3. Let $\mathcal{P}$ be the set of all planets in the universe. Let d be the function that takes as input a planet $P \in \mathcal{P}$ and gives as output the diameter of P in kilometers.

   (a) Determine a valid codomain for d. Justify your answer.

   (b) Determine the range of d. Justify your answer.

   (c) Explain why d is a well-defined function from $\mathcal{P}$ to the codomain you gave in part (a).

4. Let $\mathcal{W}$ be the set of all words in the English language and let $\mathcal{L}$ be the set of letters $\{\text{A}, \text{B}, \text{C}, \ldots, \text{Z}\}$. Let $f : \mathcal{W} \to \mathcal{L}$ be the function that takes as input a word w and outputs the first letter of the word.

   (a) Compute $f(\{\text{APPLE}, \text{CHERRY}, \text{STRAWBERRY}\})$.

   (b) Give **three** elements in the set $f^{-1}(\text{P})$.

   (c) Is f a **bijection**? Use one or two sentences to explain your answer.

Page Score

5. (a) Rewrite $\frac{3^n \cdot 6^{2n}}{2^n}$ so that it is of the form $(\text{something})^{\text{something else}}$.

(b) Compute $\log_2(2^{35})$.

(c) Rewrite $\ln(x+y)$ as an expression involving $\log_{10}$.

6. Write the following expression in product (Π) notation.

$$3^2 \cdot 4^3 \cdot 5^4 \cdot \dots \cdot 43^{42}$$

7. Compute the value of $1 + 6 + 11 + 16 + 21 + \ldots + 331$. Show your work.

8. Rewrite $\log_2\left(\prod_{i=10}^{100} i^4\right)$ as an expression involving summation notation.

Page Score

9. Recall that a deck of cards has fifty-two cards: four suits (Hearts, Diamonds, Spades, and Clubs) each with thirteen denominations (Ace, King, Queen, Jack, 10, 9, 8, 7, 6, 5, 4, 3, 2).

Hearts:	A♥ K♥ Q♥ J♥ 10♥ 9♥ 8♥ 7♥ 6♥ 5♥ 4♥ 3♥ 2♥
Diamonds:	A♦ K♦ Q♦ J♦ 10♦ 9♦ 8♦ 7♦ 6♦ 5♦ 4♦ 3♦ 2♦
Spades:	A♠ K♠ Q♠ J♠ 10♠ 9♠ 8♠ 7♠ 6♠ 5♦ 4♠ 3♠ 2♠
Clubs:	A♣ K♣ Q♣ J♣ 10♣ 9♣ 8♣ 7♣ 6♣ 5♣ 4♣ 3♣ 2♣

(a) How many different orderings are there of the entire deck of cards?

(b) In how many ways are there to have an (unordered) hand of seven cards, **exactly four** of which are hearts?

(c) How many five-card hands have at least one King and at least one Spade?
[*Hint: Draw a Venn Diagram.*]

10. Your cousin loves legos. She has legos that are all the same shape and look identical except that they come in three colors: red, blue, and green.

(a) In how many ways can she build a tower using exactly four blue legos and eleven red legos? Include one or two sentences to explain your answer.
(The tower is built up from the ground by placing legos one on top of the other.)

(b) In how many different ways can she give you a bag of 20 red, blue, and green legos to take home, with **at least one** of each color? Include one or two sentences to explain your answer.

Page Score

11. (a) Give the prime factorizations of both 144 and 1200.

(b) Use your answers from part (a) to find $\text{lcm}(144, 1200)$.

12. Use the Extended Euclidean Algorithm to find the multiplicative inverse of 23 mod 111.

13. In $\mathbb{Z}_{26}$, it is true that $3 \cdot 9 \equiv 1$, $5 \cdot 21 \equiv 1$, $7 \cdot 15 \equiv 1$, $11 \cdot 19 \equiv 1$, and $17 \cdot 23 \equiv 1 \bmod 26$.

Use this information to solve $15x + 3 \equiv 7 \bmod 26$.

[*Your answer should be of the form* $x \equiv a \bmod 26$ *for* $a \in \mathbb{Z}_{26}$.]

14. When you apply the Chinese Remainder Theorem to solve the system below, you must find the values of x_1, x_2, and x_3 by solving three congruences. What are the three congruences that must be solved?

$$\begin{aligned} x &\equiv 3 \bmod 5 \\ x &\equiv 2 \bmod 7 \\ x &\equiv -1 \bmod 11 \end{aligned}$$

[*IMPORTANT:* **DO NOT** solve the congruences. **DO NOT** find the solution to the system.]

Page Score