

QUEENS COLLEGE
Department of Mathematics
Final Examination
2½ Hours

Mathematics 151

Fall 2023

Instructions: Answer each question. Show your work and justify your answers. Partial credit will be awarded for relevant work.

1. Find the following limits.

(a) $\lim_{x \rightarrow \frac{1}{2}} \frac{x^3 - \frac{1}{8}}{1 - 2x}$

(b) $\lim_{x \rightarrow 2^-} \left(\frac{1}{|x - 2|} + \frac{1}{x - 2} \right)$

(c) $\lim_{x \rightarrow \frac{\pi}{4}} \frac{\sin(2x)}{x}$

2. (a) Find the derivative $f'(x)$ of the function $f(x) = \sqrt{2x + 3}$ using only the definition of the derivative.
(b) Find the following limit or explain why it does not exist:

$$\lim_{x \rightarrow 1} (x - 1) \sin(1/|x - 1|).$$

3. Find the following limits. If the limit is infinite, indicate whether it is ∞ or $-\infty$ and write DNE if the limit does not exist. Give an explanation for your conclusions.

(a) $\lim_{x \rightarrow 2^+} \frac{\sqrt{x - 2}}{\sin(x - 2)}$

(b) $\lim_{x \rightarrow \infty} \frac{-3x^3 + 10x^2 + 2}{-3 + 10x + 2x^3}$

(c) $\lim_{x \rightarrow \infty} (x^3 - 10000x^2)$

4. (a) State the Intermediate Value Theorem.
(b) Prove that the equation $\cos x = x^2$ has at least one real root.

5. Differentiate the following functions: (You need not simplify.)

(a) $f(x) = \sqrt{\frac{x - 1}{x + 1}}$

(b) $y = \tan^2(3x + x^2)$

(c) $g(x) = \sin \frac{1}{x}$

(d) $h(x) = \frac{1}{\sin x}$

6. (a) Find an equation of the tangent line to the curve $(1 + x) \cos(1 + y) = \cos 1$ at $(0, 0)$.
(b) The radius of a circular disk is 20 cm with maximum error in measurement of 0.3 cm. Use differentials to estimate the maximum error in the calculated area of the disk.

(continued on the back)

7. (a) Find the absolute maximum and the absolute minimum values of the function $f(x) = \frac{x}{x^2 + 4}$ when x is in the interval $[-3, -1]$.
- (b) State the theorem you are using in part a) and explain why $f(x)$ satisfies the assumptions of the theorem.

8. Find the length of the shortest line segment through the point $(1, 3)$ that is cut off by the first quadrant.

9. Integrate:

- (a) $\int x^{-2} \sin(x^{-1}) dx$
- (b) $\int \frac{dx}{(1 + 2x)^2}$
- (c) $\int_1^2 \frac{x^4 - 2x^2 + 1}{x^3} dx$
- (d) $\int_0^{\frac{\pi}{4}} \sec^2 x dx$

10. (a) Use the Midpoint rule with $n = 4$ to estimate $\int_{-1}^1 \sin^3 x dx$.
- (b) Use **only the definition** of definite integral to find $\int_0^1 (2x - 1) dx$.

$$\text{Hint: } \sum_{i=1}^n i^3 = \frac{n^2(n+1)^2}{4}, \quad \sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6} \quad \text{and} \quad \sum_{i=1}^n i = \frac{n(n+1)}{2}$$