QUEENS COLLEGE
Department of Mathematics
Final Examination
$2\frac{1}{2}$ Hours

Mathematics 152 Fall 2023

Instructions. Answer each question. Show your work and justify your answers. Partial credit will be awarded for relevant work. Calculator permitted.

Some useful trigonometric identities:

$$\sin^2\theta + \cos^2\theta = 1, \qquad \sin^2\theta = \tfrac{1}{2}(1-\cos 2\theta), \qquad \cos^2\theta = \tfrac{1}{2}(1+\cos 2\theta)$$

$$\sin 2\theta = 2\sin\theta\cos\theta, \qquad \cos 2\theta = \cos^2\theta - \sin^2\theta.$$

1. Let $f\colon \mathbb{R} \to \mathbb{R}$ by $f(x) = x^5 + 2x + 1$. (You may assume that f is surjective, i.e., onto.)
   a. Prove that f is increasing.
   b. Justify the following statement: there exists an inverse function $g = f^{-1}\colon \mathbb{R} \to \mathbb{R}$.
   c. Observe that $f(1) = 4$ and find the numerical value of the derivative $g'(4)$.

2. Find the derivative $\frac{dy}{dx}$ for each of the following. Please make basic simplifications.

   a. $y = \ln\left(\dfrac{(x^4+1)^3}{(x^3+1)^4}\right)$ b. $y = e^{2x}\,\sin^{-1}(e^{-x})$ with $x \geq 0$

3. Find each of the following indefinite integrals, well-defined up to a constant.

   a. $\displaystyle\int \frac{x^2}{\sqrt{9-x^2}}\,dx$ b. $\displaystyle\int \frac{20}{(x+3)(x^2+1)}\,dx$ c. $\displaystyle\int 5t^4\,\ln(t)\,dt$

4. Let R be the region in the *first quadrant* enclosed by the graphs of $y = 10 + 50x - 3x^3$ and $y = 10 + 2x$. Note that neither the x-axis nor the y-axis is part of the boundary.
   a. Sketch the region R, indicating points of intersection. Calculator permitted.
   b. Set up definite integrals to represent each of the following. Include limits of integration, but do not anti-differentiate or evaluate.
      i. Area of R.
      ii. Volume of the solid obtained by revolving R around the y-axis.
      iii. Volume of the solid obtained by revolving R around the line $y = -2$.
   c. Set up a definite integral to represent the length of the curve $y = 10 + 50x - 3x^3$ from the point (0,10) to the point (2,86). Approximate the numerical value of this length to two decimal places. Calculator permitted.

(continued on back)

5a. Let $a_n = \left(1 - \frac{3}{n}\right)^n$.

i. Find $\lim_{n\to\infty} a_n$, the limit of the *sequence* $\{a_n\}$ as $n \to \infty$.

ii. Does the *series* $\sum_{n=1}^{\infty} a_n$ converge or diverge? Justify your answer.

b. Let $b_n = \sqrt{\frac{n}{n^3+4}}$.

i. Prove that the series $\sum_{n=1}^{\infty} (-1)^n b_n$ converges.

ii. Is the series $\sum_{n=1}^{\infty} (-1)^n b_n$ absolutely convergent? Justify your answer.

6. Let $f(x)$ be the power series $\sum_{n=0}^{\infty} \frac{1}{2^n}(x-4)^n = 1 + \frac{1}{2}(x-4) + \frac{1}{4}(x-4)^2 + \frac{1}{8}(x-4)^3 + \dots$

a. Show that the given power series is a geometric series and add it up in closed form, to represent $f(x)$ as the ratio of polynomials.

b. For what values of x does the given power series $f(x)$ converge?

c. Find a power series for the derivative f' of f with the same center. What is the radius of convergence of this new power series?

7. Let $T_2(x)$ be the Maclaurin polynomial of degree 2 (i.e., the Taylor polynomial of degree 2 with center at 0) for the function $F(x) = \sqrt{9+2x}$.

a. Find $T_2(x)$.

b. Prove that if $x \geq 0$, then $|F(x) - T_2(x)| \leq \frac{1}{486}x^3$.

8a. Write the Maclaurin series for the exponential function e^x. If you know this series, you do not have to derive it from general theory.

b. Find the Maclaurin series for $e^{-\frac{1}{2}x^2}$.

c. Use your answer to part b to find a series that converges to $\int_0^1 e^{-\frac{1}{2}x^2}\,dx$.

d. Find the fifth partial sum s_5 of the series in part c. Calculator permitted.

e. Use the error term for alternating series to find a bound for the error

$$\left|\int_0^1 e^{-\frac{1}{2}x^2}\,dx - s_5\right|.$$